Chapter 4

Structure Theory

The goal of this last chapter is going to be to decompose connected Lie groups into "easier" building blocks. The key notions we are going to focus on are those of solvable, nilpotent, and semisimple Lie groups. We will work both at the level of Lie algebras and at the level of Lie groups.

4.1 Solvable Lie groups and Lie algebras

Definition 4.1

A group G is solvable if there exists a sequence of subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = \{e\}$$

with G_i / G_{i+1} abelian $\forall\, 0 \leq i \leq r-1$.

In studying solvable groups the derived series of a group G plays an important role.

Let $G^{(1)}$ be the subgroup of G generated by the set $\{[x,y] : x,y \in G\}$ of commutators.

Definition 4.2

The derived series of a group G is defined inductively by

$$G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$$

↳ group generated by commutators.

We have the following

Lemma 4.3

1) Let $\pi : G \to H$ be a homomorphism. Then
$$\pi(G^{(i)}) = (\pi(G))^{(i)} \quad , \quad i \geq 1.$$

2) Let $N \triangleleft G$. Then G/N is abelian iff $N \supset G^{(1)}$.

Proof

1) is clear since $\pi([x,y]) = [\pi(x), \pi(y)]$

2) Let $\pi : G \to G/N$ be the canonical

projection. Then from 1) we get:

$$[G/N, G/N] = [\pi(G), \pi(G)] = \pi([G,G])$$

Hence G/N is abelian iff

$$[G,G] \subset \operatorname{Ker} \pi = N. \qquad \square$$

$\boxed{\text{Lemma 4.4}}$

G is solvable iff $\exists\, r \geq 1$ s.t. $G^{(r)} = \{e\}$.

Proof

($\Leftarrow$) It is clear that $G^{(i)} \triangleleft G^{(i-1)}$ and by Lemma 4.3 2) $G^{(i-1)}/G^{(i)}$ is abelian. This proves that if the derived series terminates in finite steps then the group is solvable.

($\Rightarrow$) Let $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_r = \{e\}$ be as in the definition of solvability.

Claim: $G_i \supset G^{(i)}$.

We prove the claim by recurrence.
G/G_1 is abelian, hence $G_1 \supset G^{(1)}$.

If $G_{i-1} \overset{(*)}{\supset} G^{(i-1)}$, then since G_{i-1}/G_i is abelian, we have

$$G_i \underset{\text{Lemma 4.3 2)}}{\supset} G_{i-1}^{(1)} \underset{(*)}{\supset} \left(G^{(i-1)}\right)^{(1)} = G^{(i)}.$$

$\square$

Definition 4.5

Let G be a solvable group. The solvability length of G is

$$\mathrm{sol}(G) := \min\{r \geq 1 : G^{(r)} = \{e\}\}.$$

Lemma 4.6

Let $N \triangleleft G$ and $N \supset G^{(s-1)}$. Then $\mathrm{sol}(G/N) \leq s-1$.

Proof

Let as usual $\pi : G \longrightarrow G/N$ denote the canonical projection. Then we have

$$\left(G/N\right)^{(s-1)} = (\pi(G))^{(s-1)} \underset{\text{Lemma 4.3 1).}}{=} \pi(G^{(s-1)}) \underset{N \supset G^{(s-1)}}{=} \{e\}.$$

$\square$

The discussion above was about abstract groups with no extra structure. We shall see now that for Hausdorff top. groups that are solvable the subgroups in Definition 4.1 can be chosen with additional properties.

Proposition 4.7

Let G be a solvable topological Hausdorff group. Then there exists a sequence

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_r = \{e\}$$

with G_{i-1}/G_i abelian $1 \leq i \leq r-1$ and G_i closed. If G is connected the G_i's can be taken connected.

Exercise 4.8

Let G be a top. group. Let $N \lhd G$ be a closed subgroup. If N and G/N are connected then so is G.

$\boxed{\text{Lemma } 4.9}$

If G is a connected top. group then $G^{(i)}$ is connected $\forall i \geq 1$.

Proof

The set $V := \{[x,y] : x,y \in G\}$ is connected and so are all the $V^n = V \cdots V$.
But $G^{(1)} = \bigcup_{n \geq 1} V^n$ and $V^n \ni e$ $\forall n \geq 1$.
The conclusion follows for $i=1$.
The argument for general $i \geq 1$ is analogous. $\square$

Proof of Proposition 4.7

The proof is by induction on $sol(G) =: r$.

If $r=1$ then $G^{(1)} = \{e\}$ and $\{e\}$ is closed.

Let $r \geq 2$. $G^{(r-1)}$ is abelian and normal in G. Hence $\overline{G^{(r-1)}}$ also is abelian and normal in G. Moreover $sol\left(G/\overline{G^{(r-1)}}\right) \leq r-1$ and $G/\overline{G^{(r-1)}}$ is a Hausdorff top. group.

Now we can consider, $\pi : G \longrightarrow G/\overline{G^{(r-1)}}$

and apply the inductive hypothesis to obtain

$$G/\overline{G^{(r-1)}} = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_e = \{e\}$$

with H_i closed and H_{i-1}/H_i abelian.

Then

$$G = \pi^{-1}(H_0) \triangleright \pi^{-1}(H_1) \triangleright \cdots \triangleright \pi^{-1}(H_e) = \overline{G^{(r-1)}}$$

does the job (Check this!)

If G is in addition connected then $\overline{G^{(r-1)}}$ is connected. Then one can take the H_i's connected and use Exercise 4.8 to conclude. $\square$

Corollary 4.10

Let G be a connected solvable Lie group. Then there exists a sequence

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = \{e\}$$

where G_i is closed connected and G_{i-1}/G_i is isomorphic to either $\mathbb{R}$ or $\mathbb{T}$ for $1 \leq i \leq r-1$.

Proof

Combine Proposition 4.7 with the classification of connected abelian Lie groups. Sheet 5 Exercise 2. $\square$

$\boxed{\text{Example 4.11}}$

$$G = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & & \ddots \\ & & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} : 1 \in \mathbb{R}^*, * \in \mathbb{R} \right\}$$

$$\subseteq GL(n,\mathbb{R}).$$

$$G^{(1)} = \left\{ \begin{pmatrix} 1 & \ddots & * \\ 0 & \ddots & \ddots \\ & & 1 \end{pmatrix} : * \in \mathbb{R} \right\}$$

$$G^{(2)} = \left\{ \begin{pmatrix} 1 & 0 & \ddots & * \\ & \ddots & \ddots & \\ & 0 & \ddots & 0 \\ & & & 1 \end{pmatrix} : * \in \mathbb{R} \right\}.$$

$$G^{(n)} = \{ 1\!\!1 \}.$$

This is the prototype of solvable Lie group.

Indeed we have the first fundamental theorem of Lie.

Theorem 4.12 [Lie's theorem]
Let G be a connected Lie group that is solvable as a group and $\rho : G \longrightarrow GL(V)$ be a representation into a complex vector space V. Then there is a basis of V such that $\rho(g)$ is upper triangular $\forall g \in G$.

Definition 4.13
Let G be a Lie group and $\rho : G \longrightarrow GL(V)$ be a complex representation. A weight of G in V is a homomorphism

$$\chi : G \longrightarrow \mathbb{C}^*$$

such that $V_\chi := \{ v \in V : \rho(g) v = \chi(g) v \ \forall v \in V \} \neq \{0\}$

If χ is a weight then V_χ is the weight space and any $v \in V_\chi \setminus \{0\}$ is a weight vector.

Remark 4.14
A weight χ is a smooth homomorphism.

Theorem 4.15

Let G be a connected Lie group that is solvable and $\rho: G \longrightarrow GL(V)$ be a complex representation. Then G has a weight in V.

The proof of Theorem 4.15 relies on the following

Lemma 4.16

Let G be a connected Lie group and $\rho: G \longrightarrow GL(V)$ be a complex representation. Let $H \trianglelefteq G$ and $\chi: H \longrightarrow \mathbb{C}^*$ be a weight of H in $\rho|_H: H \longrightarrow GL(V)$. Then V_χ is $\rho(G)$-invariant.

Proof

Given $g \in G$, $h \in H$, $v \in V_\chi$ we have.

$$\rho(h)\rho(g)v = \rho(g)\rho(g^{-1}hg)v = \chi(g^{-1}hg)\rho(g)v.$$

Now $\chi(g^{-1}hg) \in \mathrm{Spec}(\rho(h)) \subset \mathbb{C}^*$

Thus we get a continuous map

$$\begin{aligned} G &\longrightarrow \mathrm{Spec}(\rho(h)) \\ g &\longmapsto \chi(g^{-1}hg). \end{aligned}$$

Since G is connected and $\mathrm{Spec}(\rho(h))$ is finite, the map is constant. Hence $\chi(g^{-1}hg) = \chi(h)$ $\forall g \in G$ $\forall h \in H$.

Therefore we conclude that $\rho(g) V_\chi \subset V_\chi$. $\square$

Proof of Theorem 4.15

The proof is by induction on $\dim G$.

If $\dim G = 1$ then $\dim \mathfrak{g} = 1$ so $\mathfrak{g} = \mathbb{R}X$ for some $X \in \mathfrak{g}$.

Let $v_0 \neq 0$ in V be an eigenvector of $d\rho(X)$. Then $d\rho(Y)\mathbb{C}v \subset \mathbb{C}v$ $\forall Y \in \mathfrak{g}$ and since G is connected.

by Proposition 3.101 we infer that
$\rho(G)\,\mathbb{C}v \subset \rho(G)\,\mathbb{C}v$. Hence
$\rho(g)v = \chi(g)v$ and χ is a weight.

Let $\dim G \geq 2$. Let $H \trianglelefteq G$ be closed
connected normal with $G/H \cong \mathbb{T}$ or $\mathbb{R}$

By the inductive hypothesis H has a weight
$\chi : H \longrightarrow \mathbb{C}^*$ in V and by Lemma 4.16
V_χ is $\rho(G)$-invariant.

From $\rho(h)v = \chi(h)v$, $v \in V_\chi$, $h \in H$
we deduce:

$$D_e\rho(X)v = (D_e\chi)(X)\,v \quad \forall\, X \in \mathfrak{h}.$$

We can write $\mathfrak{g} = \mathbb{R}Y \oplus \mathfrak{h}$, for some
$Y \in \mathfrak{g}$ and let $v_0 \in V_\chi \setminus \{0\}$ be
an eigenvector of $d\rho(Y)$.

It follows that

$$d\rho(Z)\,\mathbb{C}v_0 \subset \mathbb{C}v_0 \quad \forall\, Z \in \mathfrak{g}.$$

By connectedness of G we get:

$$\rho(g)\,\mathbb{C}v_0 = \mathbb{C}v_0 \quad \forall g \in G$$

and hence G has a weight in V. $\square$

Use Prop 3.151 again.

Proof of Theorem 4.12

The proof will be by induction on $\dim V$.

Let $\chi : G \longrightarrow \mathbb{C}^*$ be a weight of ρ obtained by Theorem 4.15 and let V_χ be the corresponding weight space. Then: $\dim(V/V_\chi) < \dim V$ and we can obtain a representation of G in V/V_χ by setting

$$\tilde{\rho}(g)(v + V_\chi) := \rho(g)v + V_\chi .$$

Let $f := \dim V$ and $e_1, \dots, e_f$ be a basis of V_χ and $e_{f+1}, \dots, e_n \in V$ be such that: $\bar{e}_\ell := e_\ell + V_\chi \quad f+1 \leq \ell \leq n$ form a basis of V/V_χ with respect to which $\tilde{\rho}(g)$ is upper triangular $\forall g \in G$.

The last condition amounts to say that

$$\bar{\rho}(g)\bar{e}_i = \chi_i(g)\bar{e}_i + \sum_{j=f+1}^{i-1} \bar{\rho}(g)_{ji}\,\bar{e}_j$$

for any $f+1 \le i \le n$.

Hence:

$$\rho(g)e_i = \chi_i(g)e_i + \sum_{j=f+1}^{i-1} \bar{\rho}(g)_{ji}\,e_j \pmod{V_\chi}$$

That is to say $\rho(g)$ can be written as:

$$\rho(g) = \begin{array}{cc} & \overset{f}{} \\ f & \left(\begin{array}{c|c} \begin{matrix}\chi(g) & & \\ & \ddots & \\ & & \chi(g)\end{matrix} & \begin{matrix} * & \cdots & * \\ * & * & * \\ * & * & *\end{matrix} \\ \hline & \begin{matrix}\chi_{f+1}(g) & & \bar{\rho}(g)_{ji} \\ & \ddots & \\ & & \chi_n(g)\end{matrix}\end{array}\right) \end{array}$$

with respect to the basis $\{e_i\}$ of V. $\square$

We next turn to a characterization of Lie groups that are solvable in terms of their Lie algebras

Definition 4.17

A Lie algebra $\mathfrak{g}$ is solvable if there exists a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \triangleright \mathfrak{g}_1 \triangleright \cdots \quad \mathfrak{g}_r = \{0\}$$

where $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is abelian.

Example 4.18

A prototypical example of solvable Lie algebra is

$$\mathfrak{g} = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} : * \in \mathbb{R} \right\}.$$

As in the case of groups we define

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{linear span of } \{[x,y] : x, y \in \mathfrak{g}\}.$$

Definition 4.19

The derived series of a Lie algebra $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(\ell)} := \left(\mathfrak{g}^{(\ell-1)}\right)^{(1)} = [\mathfrak{g}^{(\ell-1)}, \mathfrak{g}^{(\ell-1)}], \ \ell \geq 2.$$

Definition 4.20

Let $\mathfrak{g}$ be a Lie algebra. An ideal $\mathfrak{h}$ of $\mathfrak{g}$ is characteristic if $\delta(\mathfrak{h}) \subset \mathfrak{h}$ for every derivation $\delta \in \mathrm{Der}(\mathfrak{g})$.

Lemma 4.21

If $\mathfrak{i} \subset \mathfrak{g}$ is an ideal and $\mathfrak{h} \subset \mathfrak{i}$ is a characteristic ideal in $\mathfrak{i}$ then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

__Proof__

By the Jacobi identity if $x \in \mathfrak{g}$ then the endomorphism $\delta_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined as $\delta_x(Y) := [x, Y]$ is a derivation of $\mathfrak{g}$.

Since $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, $\delta_x(\mathfrak{i}) \subset \mathfrak{i}$ and hence $\delta_x \in \mathrm{Der}(\mathfrak{i})$.

Since $\mathfrak{h}$ is characteristic in $\mathfrak{i}$

$\delta_x(Y) = [x, Y] \in \mathfrak{h} \ \ \forall \ Y \in \mathfrak{h}$, thus.

h is an ideal in $\mathfrak{g}$. $\square$

Corollary 4.22

For every $i \geq 0$ $\mathfrak{g}^{(i+1)}$ is a characteristic ideal in $\mathfrak{g}^{(i)}$ and hence an ideal in $\mathfrak{g}$.

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As in the case of groups we now have:

Corollary 4.23

1) If $\pi : \mathfrak{g} \longrightarrow h$ is a Lie algebra homomorphism we have

$$\pi\left(\mathfrak{g}^{(i)}\right) = \pi(\mathfrak{g})^{(i)} \qquad \forall\, i \geq 1.$$

2) Let $n \trianglelefteq \mathfrak{g}$. Then $\mathfrak{g}/n$ is abelian iff $n \supset \mathfrak{g}^{(1)}$.

Proof

1) follows from $\pi([x,y]) = [\pi(x), \pi(y)]$.

2) Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/n$ be the canonical projection homomorphism. Then by 1)

$$[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] = [\pi(\mathfrak{g}), \pi(\mathfrak{g})] = \pi(\mathfrak{g}^{(1)}).$$

Hence $\mathfrak{g}/\mathfrak{h}$ is abelian iff $\mathfrak{h} \supset \mathfrak{g}^{(1)}$. $\square$

$\boxed{\text{Lemma 4.24}}$

$\mathfrak{g}$ is solvable iff $\mathfrak{g}^{(r)} = 0$ for some $r \geq 1$.

<u>Proof</u>

$(\Leftarrow)$ Let us consider

$$\mathfrak{g} = \mathfrak{g}^{(0)} \triangleright \mathfrak{g}^{(1)} \triangleright \cdots \triangleright \mathfrak{g}^{(r)} = \{0\}.$$

Since $\left(\mathfrak{g}^{(i-1)}\right)^{(1)} = \mathfrak{g}^{(i)}$ by Lemma 4.23 2) we have that $\mathfrak{g}^{(i-1)}/\mathfrak{g}^{(i)}$ is abelian. Hence $\mathfrak{g}$ is solvable.

$(\Rightarrow)$ Let $\mathfrak{g} = \mathfrak{g}_0 \triangleright \cdots \triangleright \mathfrak{g}_r = \{0\}$ be such that $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is abelian for $1 \leq i \leq r$.

Since $\mathfrak{g}/\mathfrak{g}_1$ is abelian we have $\mathfrak{g}_1 \supset \mathfrak{g}^{(1)}$ by Lemma 4.23 2). Arguing inductively from $\mathfrak{g}_{(i-1)} \supset \mathfrak{g}^{(i-1)}$ we get:

$$\mathfrak{g}_i \supset (\mathfrak{g}_{i-1})^{(1)} \supset (\mathfrak{g}^{(i-1)})^{(1)} = \mathfrak{g}^{(i)},$$

since $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is abelian. $\square$

Definition 4.25

If $\mathfrak{g}$ is a solvable Lie algebra then we define:

$$\mathrm{sol}(\mathfrak{g}) := \min\{r \geq 1 : \mathfrak{g}^{(r)} = 0\}.$$

Example 4.26

Consider $\mathfrak{g} = \left\{ \begin{pmatrix} x & & x \\ & \ddots & \\ 0 & & x \end{pmatrix} : x \in \mathbb{R} \right\}$,

$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$.

Then

$$\mathfrak{g}^{(1)} = \left\{ \begin{pmatrix} 0 & & x \\ & \ddots & \\ 0 & & 0 \end{pmatrix} : x \in \mathbb{R} \right\},$$

$$\mathfrak{g}^{(2)} = \left\{ \begin{pmatrix} 0 & 0 & & x \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & & & 0 \end{pmatrix} : x \in \mathbb{R} \right\},$$

$$\mathfrak{g}^{(n)} = \{0\}.$$

Hence $sol(\mathfrak{g}) = n$.

Gutenrom is very useful for inductions!

Lemma 4.27

1) If $\mathfrak{h} < \mathfrak{g}$ and $\mathfrak{g}$ is solvable then $\mathfrak{h}$ is solvable.

2) If $\mathfrak{h} \triangleleft \mathfrak{g}$, then $\mathfrak{g}$ is solvable iff $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Proof

1) We have $\mathfrak{h}^{(i)} < \mathfrak{g}^{(i)}$ $\forall i \geq 1$. The statement follows from Lemma 4.24.

2) Let $\pi: \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ be the canonical projection. Then

$$\pi(\mathfrak{g}^{(i)}) = (\mathfrak{g}/\mathfrak{h})^{(i)}, \quad \mathfrak{h}^{(i)} < \mathfrak{g}^{(i)}.$$

Thus if $\mathfrak{g}$ is solvable then $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are solvable.

Conversely, let $m \geq 1$ be such that

$$\{0\} = (\mathfrak{g}/\mathfrak{h})^{(m)} = \pi(\mathfrak{g}^{(m)}).$$ Then

$\mathfrak{g}^{(m)} \subset \mathfrak{h}$. and if $\mathfrak{h}^{(n)} = \{0\}$ we get
$\mathfrak{g}^{(m+n)} = \{0\}$. $\square$

The proof of Lemma 4.27 actually gives:

$\boxed{\text{Corollary 4.28}}$
If $\mathfrak{h} \trianglelefteq \mathfrak{g}$ and $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable then

$$sol(\mathfrak{g}) \leq sol(\mathfrak{h}) + sol(\mathfrak{g}/\mathfrak{h}).$$

———— o ————

Coming back to Lie groups, we have:

$\boxed{\text{Theorem 4.29}}$
Let G be a connected Lie group. Then the following are equivalent:

1). $\mathfrak{g} = Lie(G)$ is solvable;

2) G is a solvable group.

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Before we move to the proof of Theorem 4.29

We record without proof a very helpful general statement about Lie group structures on quotient groups.

Theorem 4.30

Let G be a Lie group and $H \trianglelefteq G$ be a closed normal subgroup. Then the group G/H can be endowed with a unique smooth structure which makes it into a Lie group and such that the canonical projection $\pi : G \longrightarrow G/H$ is a submersion.

In this situation, denoting by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively it holds:

$$\mathrm{Lie}(G/H) \simeq \mathfrak{g}/\mathfrak{h} .$$

Proof of Theorem 4.29

2) $\Rightarrow$ 1). Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_r = \{e\}$$

be given by Proposition 4.7 such that G_i is closed and connected and

G_{i-1}/G_i is abelian for all $1 \leq i \leq r$.

Let $\mathfrak{g}_i := \mathrm{Lie}(G_i) < \mathfrak{g}$. By Corollary 3.104 1). $\mathfrak{g}_i \triangleleft \mathfrak{g}_{i-1}$. By Theorem 4.30 $\mathfrak{g}_{i-1}/\mathfrak{g}_i = \mathrm{Lie}(G_{i-1}/G_i)$ and since G_{i-1}/G_i is abelian $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is also abelian by Proposition 3.66 1).

1) $\Longrightarrow$ 2). The proof of this implication is by induction on $\mathrm{sol}(\mathfrak{g})$.

If $\mathrm{sol}(\mathfrak{g}) = 1$ then $\mathfrak{g}$ is abelian and hence also G is, since it is connected.

Assume that $r := \mathrm{sol}(\mathfrak{g}) \geq 2$. Then $\{0\} \neq \mathfrak{g}^{(r-1)} \triangleleft \mathfrak{g}$ and $\mathfrak{g}^{(r-1)}$ is abelian.

By Proposition 3.66 2). $\exp_G : \mathfrak{g}^{(r-1)} \to G$ is a homomorphism. By Corollary 4.22 $\mathfrak{g}^{(r-1)} \triangleleft \mathfrak{g}$ is an ideal. Hence by Corollary 3.104 2) $\exp_G(\mathfrak{g}^{(r-1)})$ is normal in G.

Thus $N := \overline{\exp_G(\mathfrak{g}^{(n-1)})}$ is normal, abelian and connected.

Let $\mathfrak{n} := \mathrm{Lie}(N)$. By Corollary 3.104 again, $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is an ideal. Moreover it clearly holds $\mathfrak{n} \supset \mathfrak{g}^{(n-1)}$.

Thus $\mathrm{sol}(\mathfrak{g}/\mathfrak{n}) \leq n-1$. Since $\mathrm{Lie}(G/N) = \mathfrak{g}/\mathfrak{n}$ by Theorem 4.30 the proof can be completed by induction with an argument similar to the proof of Proposition 4.7. □

Given Theorem 4.29 it makes sense to give the following:

Definition 4.31

A connected solvable Lie group is a connected Lie group G such that $\mathfrak{g} := \mathrm{Lie}(G)$ is solvable.

With a similar strategy as in the proof of Theorem 4.12 (Lie's theorem) we can prove:

Theorem 4.32.

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of a solvable Lie algebra in a complex vector space V. Then there is a basis of V with respect to which all $\rho(x)$, $x \in \mathfrak{g}$ are upper triangular.